TM/9/911

B

I have not forgotten the promise you lately made me that you would explain to me accurately the principles of fluxions & limiting ratios — nor am I willing to forego the advantages which such a promise authorises me to claim — You said that the reasonings to which I have been accustomed on those subjects are ~~illogical~~ inaccurate

A

I did — & I wish I could briefly state to you the foundation I had for that assertion — but I fear you will not at first allow or attend to the importance of my remarks.

B

I engage that I will - provided that you assure me that you know those remarks to be important — because I am aware that you have considered the subject attentively; nor do I believe that you are in the habit of enforcing unimportant remarks.

A

Well then to begin — let me ask a few questions. ~~When you~~ Allowing quantities to be generated by motion, what do you mean by their Fluxions?

B

You'll allow me also I suppose to use the old words & symbols —

A

Certainly. — I have no quarrel with them; for I have found them very serviceable upon many occasions.

B

Let then x denote any increasing magnitude — by the fluxion of x at any time I mean the rate at which x is then increasing; or (which is the same thing) any proper measure of that rate. Or perhaps I had better say, that if two quantities be increasing, the ratio between the rates at which they are increasing at any time is the ratio between their fluxions at that time

A

And by what do you measure the Rate of increase?

B

By the magnitudes generated in a given time, if equal magnitudes be always generated in <u>any</u> equal times; i.e. if the increase be uniform; but if the increase be not uniform i.e. if unequal magnitudes are generated in equal times, the rate of increase at any proposed instant is ~~not~~ measured not by the magnitude actually generated in ~~a~~ the given time immediately succeeding that instant (for that would be too great, or too little according as the successive increments in equal times are increasing or decreasing) but by the magnitude that would have been generated in the given time, had the rate of increase continued ~~a~~ uniformly the same ~~as~~ ~~at~~ during the whole of the given time as at that ~~first~~ proposed instant; i.e by the magnitude that would have been generated in the given time if the inequality in the successive increments had thence forwards ceased.

A

Truly a most Logical explanation! I profess myself ignorant of what is meant by the rate of increase, except in that particular case, where equal magnitudes are generated in any equal times; & you inform me that it is to be measured by the magnitude that would have been generated according to a certain hypothesis; very well; if you lay down your hypothesis clearly then, I shall understand you; but what is the hypothesis? why that the rate of increase is to continue uniformly the same as it was at the proposed instant — ~~Truly~~ you direct me to measure a thing unknown in general, by a particular case of that unknown thing: ~~If I~~ As I not know, what is meant by the rate of increase, I know not ~~what~~ how to estimate what magnitude would be generated by it in any time. — & your aliter explanation is just as indecisive for you say that the said magnitude is that which would be generated provided that the inequality in the generation of the successive increments was thenceforwards to cease. Now I know not what magnitude would be generated in a given time if this inequality were to cease! For example suppose we consider the flowing quantity ~~as~~ to be a line generated by the motion of a point; then I say the inequality might cease, & the point proceed afterwards with a velocity of 20 feet in a second; or with a velocity of 30 feet in a second; or with any other velocity, if you make no further restriction.

B

Give me leave to begin again; & to take your example of a line generated by the motion of a point as a general representation of any increasing quantity, which will be legitimate, because in whatever manner a quantity increases, a line may be supposed to increase according to the same law. - You'll allow that a body in motion must have at any proposed time a particular velocity - Suppose then a body to move in a strait line - The fluxion of the line at any proposed instant is to ~~the fl~~ its fluxion at any other instant, as the velocity of the moving body at the former instant, to its velocity at the latter instant; & therefore the fluxion may be denoted by some measure of the velocity i.e. the fluxion may be denoted by the space that would have been passed over in a given time, had the ~~body velocity~~ body instead of varying in its velocity ~~after its~~

moved on uniformly after its appulse to the point, whose fluxion is inquired into, with the same velocity as at that point; which agrees with what I said before.

A

But Sr I beseech you be so good as to inform me first what you mean by the velocity of a body, whose motion is not uniform; By what do you measure it? — for ~~till~~ ~~unless~~ that point ~~is settled~~ be previously settled, your last explanation is evidently of the same class as the former, & liable to the same insurmountable objections.

B

Is that now a fair question? Have not all bodies, that are in motion, velocity?

A

I really cannot tell, till velocity is defined; but be that as it may, my question is certainly fair, & of great importance; for tis from ~~for~~ want of logical accuracy on this point that much mischief has arisen to Mathematics — hear me attentively — If your only definition of the measure of velocity be the space passed over uniformly in a given time, then the term velocity does not apply to the case of a body moving ununiformly — If you use the word in a popular sense, & say that however the motion be regulated, the velocity of one body is greater, equal or less than that of another body, according as the space passed over in any time by the former body is gr. equal, or less than that passed over in the same time by the other body. – I understand you – I should ~~be able to determine~~ know by means of such a definition when one velocity is gr^t equal, or less than another*; but I should have no rule whereby ~~to~~ determine the proportion between any two velocities; or rather without some further definition, there would be no* proportion; For even in the case of uniform motion, it is owing to the definition, that (to take an example) the velocity of a body that moves two miles in an hour is double the velocity of a body that moves one mile in an hour; if the definition ~~had been~~ in the case of uniform motion had been; "Velocities are measured by the squares of the spaces dato tempore", the former velocity would have been four

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* Mathematical writers who have treated of fluxions seem to have supposed that there is something

times as greater the other.

B

Yes! Yes! all that may be very true – but to cut the matter short – suppose a body to fall by the force of gravity – its velocity

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in the nature of velocity antecedent to its defi-
nition – that there is in any propos'd case of two bodies in motion a certain ratio between the two velocities & this ratio they set themselves to find out – The in-
quiry is similar to that, in which so many learned men formerly engag'd & to which it appears by their own confession some of them devoted months & years, – concerning the real nature of ratios & their properties ~~of ratios antecedent to any defn~~
The reader will find the absurdity of this en-
quiry exposed in a very masterly manner by the excellent Dr Barrow – had but this acute reasoner been aware of the equal necessity there

increases — or, if you please — taking any two equal times, the space in the succeeding time is always greater than that in the preceding — Now suppose the body to come to some point A — & all force of gravity thenceforwards to cease — Would not the body proceed afterwards uniformly with a certain velocity — according to the first sense of velocity? This is what is called the velocity at A, whether the force ceases or not; & is proportional to the fluxion of the line at that point.

A

This you urge with a triumphant air — but if

---

is of some defining criterion of the ratio between two velocities, how differently would ~~would~~ the 1st part of his Geometrical lectures have been written, & what a mess of confusion might perhaps have been avoided.

assure you you are deceived — Force is but a
name applied when certain motions take place
you define the Force to be constant when the
motion is of such a nature — variable when
such & such other ~~laws of~~ motions obtain — but
in the mean time there really is nothing
which essentially belongs to the enquiry but
the law of the motion — & every proposition in
natural Philosophy might be demonstrated
(tho perhaps not so concisely) without introdu-
-cing the idea of force — Now your idea
of Force in fact includes

1st That when the force ceases the body will
move uniformly — because any other motion is
what you attribute to the action of some force.

[insert 2d]

3dly That assuming any ratio whatsoever
however near it be to a ratio of equality, two con-
tiguous equal times may be taken so small

that the ratio of the spaces described in those times shall be still nearer to a ratio of equality.

3dly That taking any two equal times the space in the succeeding time is greater than that in the preceding.

4thly that the force ceasing, & the body moving forward from the point A, the space in any time will not be less than the space described in any equal preceding time, nor so great as it would have been (supposing the force had continued to act) in any succeeding time. & thus the motion of the body after its appulse to the point A is precisely limited, & its velocity determined

B

I do in part see the drift of your argument, & allow that your reasoning is just; but still you must excuse me if I suspect that these subtleties argue little against the sufficiency of the explanations we already have of the principles of Fluxions - For in the same manner as we have ~~a~~ very good practical notions of time & space, & yet if they be examined metaphysically, we should find it very difficult to exhibit them clearly, so we may have an exact idea of velocity, & yet be puzzled to define it accurately - & that men have ideas on this subject sufficiently exact, & competent for all practical purposes, may be inferred from the general agreement in the conclusions derived from the application of their principles - besides you allow that in the above case of a body falling by gravity, our notion of force does determine the velocity at any point

Well then; however it is that you suppose it to be thereby determined - take it as a definition of velocity in that case, & understand analogous definitions in other cases

A

In answer to what you observe, I remark

P That this agreement in conclusions is by no means a proof of the soundness of their reasonings — It happens in spite of confusion I will illustrate this by an example — Hobbes laid down the following absurd general definition of proportionals

& other authors have laid down others, equally vague, tho not perhaps equally absurd — Now they have all of them deduced the same propositions as Euclid has from his accurate definition

Would you thence argue that we have no need of this accurate but tedious definition of Euclid! for that our untaught notion of proportionality is sufficient — No; you know better — You know that tho the conclusions deduced from these undefin'd foundations be the same, the demonstrations are clouded — the mind is clouded — is impeded in making new discoveries, not having an accurate & distinct view of the relations of things: — I allow that from your method of reasoning a definition of velocity may be made out — otherwise to what purpose should I attempt to explain what it ought to have been defined — what I say is that you set out wrong, that you stumble about in the dark, apparently ignorant of the definitions to be drawn from your own words — Mr Hobbes shall serve as an illustration here again — suppose no other writer till now had left any thing about ~~proportionals~~ ratios - we might have made out what he meant ~~by them~~ proportionals & how he ought to have defined them, (I mean there would have been sufficient data for it) — but are we therefore to say that the legitimate definition is included in Hobbes' account? — But granting you (for argument sake) your latent definition, I accuse you of causing a most lamentable wast of precious time, of perplexing mens minds & confusing their ideas, by running into the dark (as Emerson* expresses himself - tho arguing directly against me)

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* See ~~his~~ the preface to his Treatise on Fluxions

I recommend to the reader to peruse the whole of that preface carefully, after he has read this essay the account of those things which you might have readily brought forward & shewn in the broad day light; For how many men who in intellect & science were ornaments to the world have been led astray by your confused accounts, & have remained to the last unsatisfied as to the truth & accuracy of your first principles. How many who have not perhaps impugned your logic have yet been dissatisfied with your method & taken up a general aversion to the doctrine of Fluxions! — But I am wasting my time in arguing for the advantages of clearness over obscurity — & as for those other cases you adduce — Time & Space (you might have added many more) — they are nothing but instances where the same confusion prevails points that want to be discussed & settled without

:the jargon of false metaphysics – indeed the whole philosophy of Mathematics lies in confusion – but as it is beside my present purpose to reduce this mass to order, I do purposely leave every point whose discussion is not now required just as I found it.

β

My good friend! I see you are under a strong impression of the truth & importance of what you advance; & *that* from you is a sufficient reason with me to be desirous of further discussion – You would have a low opinion of me if I assented to what you have said without conviction – & you know how difficult it is for a mind enveloped in error to disentangle itself at once – You are aware also that it is possible for the defendant to ~~be in the right~~ have truth on his side, & yet be unable to answer his opponents arguments - therefore you will not exclaim as men of strong but uncultivated minds sometimes have done "Is what I ~~say~~ advance demonstration or not - if it be not, shew ~~me~~ where & why it is not - if it be, profess yourself convinced - but I wish you to proceed, & give a full explanation of velocity; & the mists of error, if error I be in, will I hope gradually clear up - so apply your torch of truth.

A

I will - & I wish my abilities would permit me to apply it in the most favorable manner - but much of its effect will I fear be for some time lost through my unskillfulness.

B

A truce with your modesty, if you would have a truce with compliments - begin I say. —

A

I obey.

Of velocity in uniform motions

Defn. A Motion is said to be uniform, when the spaces described in any two times are in the same ratio as those times. Cor. In uniform motion equal spaces are described in equal times * If this Cor was made the defn the defn would follow
equal spaces are described in any equal times.

Velocity in uniform motion is measured by the space passed over in some fixt portion of time — this fixt portion may be any whatsoever, but must be taken the same in comparing two velocities; let it be denoted by A; then the space in time A is the measure of the velocity

N.B. To avoid circumlocution I shall in future assume the velocity to be equal to its measure.

Let S denote the space passed over in the time T, & let V denote the velocity; then $V = S \times \frac{A}{T}$ *

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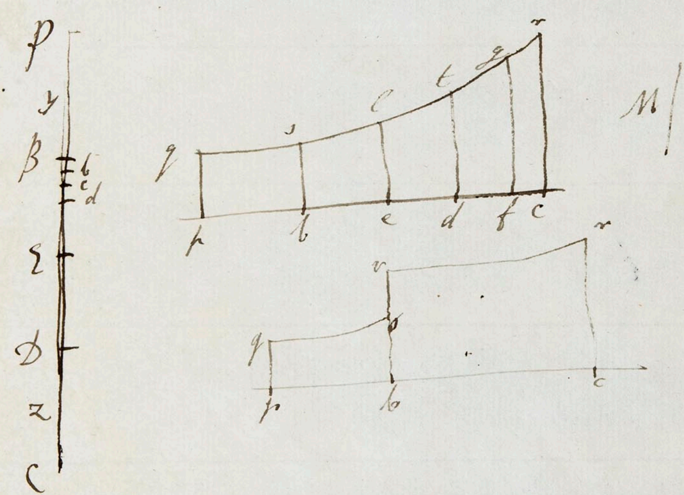
* By the expression $\frac{A}{T}$ is meant that number, whole or fractional, which denominates what multiple part or parts A is of T. e.g. if A be double of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ie. the space passed over in the time A is

$S \times \frac{A}{T}$.

For let $T = \frac{m}{n} \times A$; then since it follows from the definition that in one n^{th} of the time A the space is an n^{th} of the space in the time A; & that in m n^{ths} of the time A the space is m times the space in one n^{th}; the space in m n^{ths} of A is $\frac{m}{n}$ × space in A

ie $S = \frac{m}{n} \times V$; or $V = \frac{S}{\frac{m}{n}} = S \times \frac{A}{\frac{m}{n}A} = S \times \frac{A}{T}$.

T, $\frac{A}{T}$ denotes 2; If A be one third of T $\frac{A}{T}$ denotes $\frac{1}{3}$; in general if A be to T in the same ratio as the number ~~m~~ to the number ~~p~~ n, $\frac{A}{T}$ denotes the number ~~q~~ $\frac{m}{n}$.

a n^m pl & A T A S

The same thing may be shewn with less formality as follows

If T be equal to A, V ~~A~~ evidently equals $S \times \frac{A}{T}$;
now if we double T we double S, but we half $\frac{A}{T}$ so that the value of $S \times \frac{A}{T}$ will remain unchanged; & in the same manner it appears that in what ever ratio we change T, S will be changed in the same ratio, but $\frac{A}{T}$ will be changed in the inverse ratio; so that $S \times \frac{A}{T}$ will remain unchanged. —

Vel in dif unif motn as S

Of velocity in Variable motions

Defn Variable motion includes all that are not uniform.

The variable motions which occur in mathematics are included under the three following laws

1st Where the spaces in equal successive portions of time perpetually increase - In which case the motion is said to be accellerated.

2 Where the spaces in equal successive portions of time perpetually decrease - In which case the motion is said to be retarded.

3 Where the spaces in equal successive portions of time perpetually increase during certain times, & perpetually decrease during other times - in which case the motion is sometimes accellerated & sometimes retarded. *

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* Motions may be supposed which can be classed under none of these three laws. For Example Let a body move over a space S in a time T according to the following hypothesis. That when T is divided into two equal parts the two successive spaces are as 1 to 2; i.e.

But since in motions that vary according to the third law, the time may be divided into portions during which either the first or the second law obtains, it will be sufficient for our purpose to consider the proprieties of motions perpetually accellerated or perpetually retarded. —

First let us consider accellerated motion;

Let a body set off from A (Fig. 1) & move over the strait line AC. Let S denote any part of ~~the~~

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are $\frac{1}{3}S$ & $\frac{2}{3}S$; that when T is divided into 4 equal parts, the 4 successive spaces are as the numbers 2, 1, 1, 2. i.e. are

$\frac{2}{9}S$, $\frac{1}{9}S$, $\frac{2}{9}S$, $\frac{4}{9}S$; that when

the time is divided into 8 equal parts, the

8 successive spaces are as the numbers

1, 2, 2, 1, 1, 2, 2, 1. i.e. are

$\frac{2}{27}S$, $\frac{4}{27}S$, $\frac{2}{27}S$, $\frac{1}{27}S$, $\frac{2}{27}S$, $\frac{4}{27}S$, $\frac{8}{27}S$, $\frac{4}{27}S$.

~~space~~ PC from P towards C, & T the time of describing that part; & let A denote any fixt time; then S & T being considered variable, the Expression $S \times \frac{A}{T}$ is not (as in uniform motion) of the same magnitude, whatever T be taken, but decreases when they decrease.

For if we reduce the time to one half of its former value, & thereby double the fraction $\frac{A}{T}$ by hypothesis we reduce the space to less than one half of its former value; so that on the whole we decrease the value of $S \times \frac{A}{T}$. In

---

& so on continually; the spaces alternately upon each new division being as 1, 2, 2, 1, 1, 2, &c & 2, 1, 1, 2, 2, 1, 1, 2, &c. — in this case it would be impossible to say whether the motion at any time was accellerated or retarded.

general, if we reduce the time to $\overline{m-n}$ m^{ths} of its former value, & thereby multiply $\frac{A}{T}$ by $\frac{m}{m-n}$, we reduce the space to* less than $\overline{m-n}$ m^{ths} of its former value, or divide S by a number greater than $\frac{m}{m-n}$, & consequently on the whole diminish the value of $S \times \frac{A}{T}$. Now this decrease of the expression

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* For dividing the time as first taken into m equal portions, if the space in the first $m-n$ of these portions could possibly be so great as $\overline{m-n}$ m^{ths} of the whole space, ~~some one at least~~ the space in some one at least of these $m-n$ portions must be as great as one m^{th} of the whole space; & in each of the remaining n portions which make up the whole time, the space must be greater than in any one of the preceding portions, therefore in

$S \times \frac{A}{T}$ may either be limited or not; if it be limited, then ~~th~~ the limit of its value is called the velocity of the body at p; but if by decreasing T $S \times \frac{A}{T}$ decreases sine limite, the velocity of the body at p is said to be 0.

A note. By the limit of $S \times \frac{A}{T}$ is meant that value (V) to which $S \times \frac{A}{T}$ can by diminishing T approach sine limite, but which it ~~can~~ never exceeds; it is therefore

---

each of the n remaining portions the space must be greater than one m^{th} of the whole space & the sum of the spaces in all these n portions must be greater than n m^{ths} of the whole space; since then the ~~sum of the~~ space in the first $m-n$ portions is as great as $m-n$ m^{ths} of the whole space, & the space in the remaining n portions greater than n m^{ths} of the whole space, the two spaces together are greater than $m-n$ m^{ths} + n m^{ths} of the whole space; i.e. are greater than the whole space, such a value, that if any less space whatsoever $V-w$ were proposed (however small it be), $S \times \frac{A}{T}$ could by diminishing T be made less than $V-w$.

end of note

In the preceding article we considered the velocity at first setting off; Now let us suppose to have moved from p to B, & to be proceeding on towards C; & let S denote any space immediately succeeding the point B, & T the corresponding time. Then in the same manner as in the last case it may be proved that $S \times \frac{A}{T}$ the less T is taken, the less will be the value of $S \times \frac{A}{T}$.

In this case $S \times \frac{A}{T}$ will necessarily have a limit,

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but the two spaces together exactly make up the whole space, & these things are contradictory therefore it is absurd to suppose the space in the first $\overline{m-n}$ n^{ths} of the whole time so great as $\overline{m-n}$ m^{ths} of the whole space.

& that limit is called the velocity at B.

$S \times \frac{A}{T}$ will necessarily have a limit. For let t denote the time thro' PB; & let Bb be described in a time equal to $\frac{t}{n}$, ~~then~~ n & Bb being supposed variable; then if $S \times \frac{A}{T}$ had no limit $Bb \times \frac{A}{\frac{t}{n}}$ would have no limit i.e. $nBb \times \frac{A}{t}$ would have no limit & therefore nBb would have no limit less than which it could not become. but if PB be devided into n parts, each of which is described in a time equal to $\frac{t}{n}$ each of those parts will be less than Bb, because they preceede Bb & the time thro' each of them is equal to the time thro' Bb. — therefore the sum of these n parts will be less than nBb — but their sum is the determinate space PB therefore nBb has a limit. & consequently $Bb \times \frac{A}{\frac{t}{n}}$ or $S \times \frac{A}{T}$.

Lemma

If the limit of $S \times \frac{A}{T}$ be constantly the same for any point in the line YZ, YZ is described uniformly.

For if YZ be not described uniformly, there must be some two contiguous unequal spaces BE, ED described in equal times. Let t, the time of describing BE or ED be divided into n equal portions & let Bb, bc, cd &c be the spaces in those portions; & let L denote the constant limit of $S \times \frac{A}{T}$; then since the difference between L & $S \times \frac{A}{T}$ may by decreasing T be made less than any proposed quantity whatsoever, we may by increasing n have any one of the n expressions $Bb \times \frac{A}{\frac{t}{n}}$, $bc \times \frac{A}{\frac{t}{n}}$ &c more nearly in a ratio of equality with L than $1+x : 1$ or $1 : 1+x$, howsoever small x be taken — let n be taken sufficiently great for that purpose; then since

$Bb \times \frac{A}{\frac{t}{n}} : L$, $bc \times \frac{A}{\frac{t}{n}} : L$, &c are all less ratios than $1+x : 1$ or $1 : 1+x$, the sum of the antecedents will be to the* sum of the consequents in less ratio than $1+x : 1$ or $1 : 1+x$; o.r. $Bb \times \frac{A}{\frac{t}{n}} + bc \times \frac{A}{\frac{t}{n}} + $&c $: L + L +$ &c in less ratio than $1+x : 1$ or $1 : 1+x$, i.e. $BE \times \frac{A}{\frac{t}{n}} : nL$ ~~a less ratio than~~ or $BE \times \frac{A}{t} : L$ may be a less ratio of inequality than $1+x : 1$ or $1 : 1+x$ therefore since the ratio $1+x : 1$ may by decreasing

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* The ratios $Bb \times \frac{A}{\frac{t}{n}}$, $bc \times \frac{A}{\frac{t}{n}}$ &c may be all ratios of majority or all ratios of minority or some of one kind & some of the other; but which ever be the case, upon supposition that they are all less than the ratio $1+x : 1$ or the ratio $1 : 1+x$ it will follow from the common properties of ratios that the sum of the antecedents will be to the sum of the consequents in a less ratio than $1+x : 1$ or $1 : 1+x$

x be made less than any proposed ratio whatsoever, & $BE \times \frac{A}{t} : L$ is some certain first ratio, $BE \times \frac{A}{t} : L$ is a ratio of equality; in the same manner it may be proved that $ED \times \frac{A}{t} : L$ is a ratio of Equality. hence $BE = ED$ but BE was supposed unequal to ED; & these things are contradictory; therefore then the limit of $S \times \frac{A}{T}$ is constant, the spaces in equal times cannot be unequal, or in other words the motion must be uniform. — end of Lemma

The limit of $S \times \frac{A}{T}$ belonging to any point is less than the limit of $S \times \frac{A}{T}$ belonging to any succeeding point. For taking D any point between B & C, the limit belonging to D cannot be less than the limit belonging to B; because taking T the same in both expressions S will be less at B than ~~greater~~ at D; so that to whatever value $S \times \frac{A}{T}$ belonging to D can be reduced, $5x\frac{A}{7}$ belonging to B can be reduced to *less value; & the same reasoning holds for any point between D & B; therefore either the said limit belonging to D is greater than that belonging to B, or else the limits to any point between B & D are all equal to that at B; but this latter supposition is impossible by the last article; therefore the limit to D is greater than that at B.

It might appear at first sight that we might hence conclude the proposition to be true. But such reasoning would not be logical. For it ~~may~~ might happen, that taking I always the same in two cases the value of $5x\frac{A}{7}$ ~~should~~ in one case should be always less than its value in the other, & yet the limits of the two expressions be equal.

P

Thus it appears that if a body moves from ~~B~~ to C according to the hypothesis of accellerated motion, the expression $S \times \frac{A}{T}$ for any point between A & C will have a limit, & that this limit increases as the body approaches to C. Now this increase ~~s~~ may ~~be~~ either be limited or not; if it be, i.e. if some quantity can be assigned, so great as which none of the said limits can be found, but to which the said limits do approach within any degree of exactness by taking the point nearer & nearer to C, that quantity is called the last velocity or the velocity at C. But if the limits increase sine limite, i.e. if no quantity howsoever great can be assigned, but that we can find some point so near to C, that the limit belonging to it ~~that point~~ shall be greater than the assign'd quantity, the velocity at C is said to be infinite.

which vary according to the 2d law

Now let us consider ~~retarded~~ motions; Let a body move from P to C with a motion continually retarded. Still by the velocity at any point B is meant the limit of the expression $S \times \frac{A}{T}$ where S A & T denote as before; & by reasoning similar to that in the preceding articles we should find, 1st that $S \times \frac{A}{T}$ is increased by decreasing T. 2dly that for any point between P & C it must necessarily have a limit. 3dly that its limit is less for a point nearer to C than for one more remote. 4thly That at P, $S \times \frac{A}{T}$ may have no limit greater than which it cannot become; but by decreasing T may be capable of ~~being made~~ ~~becoming greater~~ ~~than~~ exceeding any proposed quantity. 5thly. That at C the end of motion it may have no limit less than which it cannot become.

In finding the ratio of the limits of $S \times \frac{A}{T}$ in two different cases we may take T to be the same in both cases Let then S & s denote two spaces in two different cases described in the same time T. Then the limiting ratio of the ratio $S \times \frac{A}{T} : s \times \frac{A}{T}$ is the same as the ratio limit of $S \times \frac{A}{T}$ to limit of $s \times \frac{A}{T}$; but the limiting ratio of $S \times \frac{A}{T} : s \times \frac{A}{T}$ is the same as the limiting ratio of $S : s$; &

limit of $S \times \frac{A}{T}$: limit of $s \times \frac{A}{T}$ is the ratio between the velocities; therefore the velocities in two cases are in the limiting ratio of the spaces described in the same time ~~& it would perhaps have~~ & if I wished to deviate as little as possible from the established methods I should give that as a definition of velocity instead of the limit of $S \times \frac{A}{T}$ — 'Tis what the writers on this subject ought to have premised as a definition.

B

You define the velocity at any point to be the limit of $S \times \frac{A}{T}$ where S & T denote the succeeding time & space. This definition does not include the terminating point of the space - is there not therefore an inaccuracy in speaking of the last velocity?

A

There certainly would be inaccuracy in speaking of last velocity without defining what is meant; but I did define it to be the limit of those limits we have been speaking of. But it would be of no consequence if we said there was no velocity at the end of motion.

I used the succeeding spaces, because the established writers have I think for the most part us'd them: But it would have been equally convenient to have employed the preceding spaces,

& have defin'd the velocity at any point to have been the limit of $S \times \frac{A}{T}$ where S denotes the space immediately preceding that point. Indeed in most of the cases that occur in the application of these principles, the limit of $S \times \frac{A}{T}$ would be the same whether S denoted the succeeding or the preceding space - & in those cases the velocity may be defined to be either limit; but in cases where the two limits are different, we could not say that the velocity is equal to either limit, unless we allowed that two velocities may belong to the same point

B

But is it not a necessary consequence of diminishing the times sine limite, that the two contiguous spaces in equal times should approach to a ratio of equality?

A

Certainly not. Unless you further restrict the definition of accellerated & retarded motion.

B

& is it easy or natural to suppose velocity to increase per saltum.?

A

As easy as to conceive any other motion. For – but first give me leave to observe, that when you are convinced of this, you will be convinced of having entertained one erroneous notion at least on this subject, for it appears from what you have just said that you did suppose that if a body moved any how from A to C it would have a certain velocity at any proposed point B & if it moved back again from C to A, with the motion reversed, i.e. in such a manner that the time of describing any particular part of the space CA should be equal to the time in which that part was described before, the velocity at the point B would necessarily be the same as before; but this is not universally true, unless you allow that two velocities may belong to the same point. But all this will appear much clearer to you if time & space be represented geometrically in the accustomed manner.

Let therefore pc denote the time of a bodies motion from P to C (fig) & let an ordinate perpendicular to pc move uniformly, parallel to itself from p to c, & let its length vary in such a manner, that the area passed over by it in any time shall be proportional to the space passed over by the body in the same time; then the ordinate will vary as the velocity i.e as the limit of $P \times \frac{A}{T}$ or Velocity at any point B will be to velocity at any other point D as bs, to dt.

For if BE, DF be described in equal times bc df, Velocity at B : velocity at D in limiting ratio of BE to DF, therefore by hypothesis in the limiting ratio of bcls to dfgt, but the limiting ratio of bels to dfgt you very well knew is $bs \times be$: $dt \times df$ (or since be always equals df) $bs : dt$ *

Now if we suppose the line gr to be nowhere perpendicular to pc (as in fig) then since the ~~limit of the area on one side of any ordinate is equal to the limit~~ limiting ratio of the area on one side of any ordinate to the area on other side (the bases being the same) is a ratio of equality, it would make no difference whether the limiting ratio of the succeeding spaces or of the preceding be taken to

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* Let the line M (fig 3) denote the time A; then $S \times \frac{A}{T}$ belonging to B will be denoted by $\frac{bels \times M}{be}$ & the limit of $S \times \frac{A}{T}$ will be denoted by $\frac{bs \times be \times M}{be}$ i.e. by $bs \times M$, the rectangle under the ordinate & the first line M.

denote the ratio between the velocities. But if we suppose the space to increase in such a manner as that the line qr shall sometimes be in a direction perpendicular to pc (as in fig. from s to v) the expression $S \times \frac{A}{T}$ may always have a limit, but at the point B corresponding to the point of time b, the limit when S denotes the succeeding space will be to the limit when S denotes the preceding space as bv to bs which may be any ratio whatsoever; so that if we consider the velocity as being denoted by both the limits, the velocity at B will be twofold; & if we consider the velocity as being denoted by the succeeding limit only or the preceding limit only, then if the body were to move back again in the same manner, the velocity at the same point would be different — yet nothing is more easy to imagine than that the space should so increase — This double value of the limit can occasion no difficulty in mathematical investigations — I mention it only with a view of shewing you what confused notions you have had about velocity; & how erroneous the sentiment is which occasions ~~me~~ you to say that there is no need of ~~accuracy in~~ defining velocity, for that every body in motion must have a certain velocity at each point of time, & all men mean the same thing. It is certainly true that men do in most cases come to the same conclusions; & this happens as I observed before in spite of confusion, for altho velocity is ridiculously at first explaind to be the space that would be passed over if &c, yet from the method of investigating the velocity in any case, it appears that the above limit does in fact coincide with it; so that whatever fantastic bewilderd & confused metaphysical notion we may at first entertain of velocity our practice tends to correct it.

B

It appears then that nothing more is wanted than a proper definition of velocity.

A

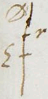
Pardon me – many more things are wanted – some to be omitted & some inserted. In the first place the proposition that attempts to prove the ratio between the velocities to be the same as the limiting ratio of the spaces dato tempore ought to be omitted – For it is either identical or unintelligible ——

B

Pray stop a moment — By the expression "Space that would be generated &c" may we not understand in accelerated motion either a velocity greater than sufficient to generate the space s in the time t, s being the space really passed over in the previous time t, but which velocity being diminished in any degree whatsoever, there might be taken a time so small, that the space really passed over in that time should be greater than the space that could be generated in it by the diminished velocity or else a velocity less than sufficient to generate the space s in the time t, s being the space really passed over in the succeeding time t, but which &c mutatis mutandis. & the like in retarded motion.— Would not this be a definite notion of velocity— lead to the same conclusions as the limit of $\int \times \frac{A}{T}$ or the limiting ratio of the spaces in equal times & yet require their agreement to be proved?

A

Your question in fact is this — May we not define the velocity at any point in accellerated motion to be greater than the velocity indicated by any preceding space but as little greater as possible – or less than the velocity indicated by any succeeding space, but as little less as possible; & the like in retarded motion. & it would exactly coincide with the definition of the velocities being in the limiting ratio of the spaces described in equal times, & involve all the consequences I have shewn to follow from it – But to what purpose is it to suppose a definition to lie hid in that manner? granting that it were implied, then, as I observed before, I object to the elementary treatises for not bringing it forward more explicitly – What a mass of – what shall I call it – stupefies the reader of MacLaurin's Introductory part – & after all leaves his notions as confused as ever –



Suppose a body to move from D to E with accellerated motion, & that Dr is described in a time t; by the words "the velocity indicated by the space Dr" I mean the velocity sufficient for the uniform description of Dr in the time t.

Yet how highly have I heard MacLaurin's ~~introductory~~ "Grounds of the method of Fluxions" commended for settling one's notions on that subject! tho' I confess I have never thorowly perused it.

Eh! I'll allow, it might have great effect in settling the mind — especially if read after supper! but seriously to recur to your question — I think it cannot be truly ~~said that~~ affirmed that the said words are intended to convey your definition of velocity, because it is* notorious that authors do beforehand either consider the meaning of the word velocity as previously settled & known, or else attempt a Metaphysical & useless definition —

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* Let any man read over that part of Ludlams Essay on ultimate ratios, which treats of the fluxion of x^n — & he will see that the author considered velocity or rate of increase as sufficiently understood without any definition, & accordingly sets himself about finding out the velocity in that particular case — Now Ludlam was a man of sound judgement in general, & is allowed (justly I think) to have had

but that you may immediately judge for your self, let us read over some of these articles from Maclaurin.

[Extracts]

I shall offer no other comment on these extracts than what I have allready remarked – I am deterred by the example of the excellent Dr Barrow, as Maclaurin justly calls him (tho unluckily quoting a passage where that excellence forsook him) who in his* admirable mathematical lectures, has,

* I by no means intend to apply this epithet to every one of those lectures – some of them are very faulty & illogical.

the start of his predecessors & cotemporaries in point of accuracy & precision – If then Ludlam, even where he was pointing out & correcting the inaccuracies of other mathematicians shews explicitly by his words that he was led away by a loose & popular notion of velocity, it is a proof that there was no correcter notions on the subject in his time, & makes good my assertions respecting Maclaurin & others – not that those assertion need such a proof – they will stand or fall according to the result of an examination of the authors alluded to – but I thought this sort of proof might be satisfactory to many without further trouble.

 ~~by attempting to expose the~~ in exposing the false logic & inaccuracies of several eminent mathematicians, involved himself in such a labarinth of words as must prevent many from reading him.

The proper method of introducing true logic in the doctrine of Fluxions is (it appears to me) not by concerning ourselves much at first about unravelling the tangles of others, but by laying down rigidly & accurately the principles of the method – for then we shall have a clearer light for perceiving where it is that others have erred – therefore if you please I will proceed to establish the principles as well as I am able – But in order to cheer you after this dry discussion, I give you notice that the demonstrations will be much easier than those cramp't theorems you have just heard about velocity – For I shall take the liberty of discarding the notions both of time & velo-city however firmly they may think themselves established – they being totally unnecessary & foreign to the subject in general, & only serving under the specious appearance at first of Brevity & preciseness, to introduce (if any thing like accuracy be attempted) tediousness, perplexity & confusion.

B

That is indeed comfortable news – since velocity must involve us in such a train of subtleties – Napier I believe was one of the first who introduced the notion of ~~abstract~~ velocity into abstract geometry – it has been much disliked by severall learned mathematicians – particularly by* Kepler ~~since the days of Newton it has met with general tho not universal approbation~~ it owes its firm establishment to Newton I suppose – by the bye if it be unnecessary, how came that great man to adopt it in his reasonings?

* See Maseres Scriptores Logarithmici Vol