TM/9/910

Of the five Spirals when $n-1=-3$ or the

Force $\propto \frac{1}{\text{Distance}^3}$

Case 1st.

Let the vel.y be that acquired by falling from the distance P & let d = the $\perp$r on the tangent at V. Then $PQRV \pm VLED = \frac{D}{-2} \times \bar{x}^{-2}$

$= \frac{D}{2} \times x^{-2} - p^{-2} = \frac{D}{2} \cdot \frac{p^2 - x^2}{p^2 x^2}$ hence $\frac{Q \times IN \times CX^2}{2A^2 \sqrt{\text{area} - Z^2}} = \frac{\frac{1}{2} r^2 Q \times \pm \dot{x}}{x^2 \sqrt{\frac{D}{2} \cdot \frac{p^2 - x^2}{p^2 x^2}} - }$

$= \frac{\frac{1}{2} r^2 Q \times \pm \dot{x}}{x \sqrt{\frac{D}{2} \cdot \frac{p^2 - x^2}{p^2} - Q^2}} = \frac{\frac{1}{2} r^2 Q \times \pm \dot{x}}{x \sqrt{\frac{D}{2p^2} \times p^2 - x^2 - \frac{2p^2 Q^2}{D}}} = \frac{\frac{\frac{1}{2} r^2 QP}{\sqrt{\frac{D}{2}}} \times \pm \dot{x}}{x \sqrt{p^2 - \frac{2p^2 Q^2}{D} - x^2}}$

$= \frac{\frac{1}{2} r^2 \sqrt{\frac{2p^2 Q^2}{D}} \times \pm \dot{x}}{x \sqrt{p^2 - \frac{2p^2 Q^2}{D} - x^2}}$, where the constant q.y $p^2 - \frac{2p^2 Q^2}{D}$ is positive, be

* D is gr than $2Q^2$ & consequently $\frac{2Q^2}{D} \times p^2$ is less than p^2.

put $p^2 - \frac{2Q^2 p^2}{D} = a^2$; then $p^2 - a^2 = \frac{2Q^2 p^2}{D}$ & $\sqrt{\frac{2p^2 Q^2}{D}} = \sqrt{p^2 - a^2}$

hence fluxion becomes $\frac{\frac{1}{2} r^2 \sqrt{p^2 - a^2} \times \pm \dot{x}}{x \sqrt{a^2 - x^2}}$, whose fluent may be

expressed both by Logarithms & by hyperbolic Sectors. By Log.ms the

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~~$2Q^2 = 2^2 \times d^2 \times \frac{p^2 - r^2}{p^2 r^2}$~~

* $2Q^2 = 2d^2 \frac{D}{2} \times \frac{p^2 - r^2}{p^2 r^2} = \frac{d^2}{r^2} \times \frac{p^2 - r^2}{r^2} \times D$ which is less than D

because ~~$\frac{d^2}{r^2}$ is less than 1 &~~ both $\frac{d^2}{r^2}$ & $\frac{p^2 - r^2}{r^2}$ are less than 1.

fluent is $\left(\int \frac{1}{2}r^2 \frac{\sqrt{p^2-a^2}}{2a} \times \frac{\pm 2ax}{x\sqrt{a^2-x^2}} + C = \right) \frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{2a} \times \pm h.l. \frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}} + C$

but $VCX = 0$ when $x = r$ $\therefore$ equation becomes $0 = \frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{2a} \times \pm h.l. \frac{a-\sqrt{a^2-r^2}}{a+\sqrt{a^2-r^2}} + C$

& $C = \mp \frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{2a}$ $\therefore VCX = \frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{2a} \times \left(h.l. \frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}} \sim h.l. \frac{a-\sqrt{a^2-r^2}}{a+\sqrt{a^2-r^2}}\right)$

The body comes to a higher apse when $x = a$, & the sector VCG between the apse distance & distance CV is $\frac{1}{2}r^2 \times \frac{\sqrt{p^2-a^2}}{2a}$ $\times \left(h.l. \frac{a-\sqrt{0}}{a+\sqrt{0}} \sim h.l. \frac{a-\sqrt{a^2-r^2}}{a+\sqrt{a^2-r^2}}\right) = \frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{2a} \times h.l. \frac{a+\sqrt{a^2-r^2}}{a-\sqrt{a^2-r^2}}$

$\therefore$ substituting for r any other distance $x = CJ$, the sector GCX $= \frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{2a} \times h.l. \frac{a-\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}} = \frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{2a} \times h.l. \frac{\overline{a-\sqrt{a^2-x^2}}^2}{a+\sqrt{a^2-x^2} \times a-\sqrt{a^2-x^2}}$

$\frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{2a} \times h.l. \frac{\overline{a-\sqrt{a^2-x^2}}^2}{x^2} = \frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{a} \times h.l. \frac{a-\sqrt{a^2-x^2}}{x} = l. \frac{a-\sqrt{a^2-x^2}}{x}$

to modulus $\frac{1}{2}r^2\frac{\sqrt{p^2-a^2}}{a}$

$\frac{2Q^2p^2}{D}$ or $\left(\frac{Qp}{\sqrt{\frac{D}{2}}}\right)^2 = \frac{2}{D}p^2 \times d^2 \times \frac{D}{2} \times \frac{p^2-r^2}{p^2r^2} = \frac{d^2}{r^2} \times \overline{p^2-r^2}$, wh

$\therefore$ is a constant qy. whatever r be. hence $a^2 w^2 = p^2 - \frac{2p^2Q^2}{D} =$

$p^2 - \frac{d^2}{r^2} \times \overline{p^2-r^2}$; & $p^2 - a^2 = \frac{d^2}{r^2} \times \overline{p^2-r^2}$ a constant qy.

Hence, if any ray CJ be produced to A (so that $CA = p$) & from A a $\perp$r AT be drawn upon the tangent at J, T will be in the periphery of the $\odot$ whose radius is a, the apse dist. i.e. if CQ be $\perp$r upon the tangent, $CQ^2 + QT^2$ will be constant.

For $CV : AV :: VQ : VT$

& Comp. $CV : CA :: VQ : QT$

& $CV^2 : VQ^2 :: CA^2 : QT^2 = \frac{CA^2 \times VQ^2}{CV^2} = p^2 \cdot \frac{\overline{r^2 - d^2}}{r^2}$

$= p^2 - \frac{p^2 d^2}{r^2} \therefore QT^2 + CQ^2 = p^2 - \frac{p^2 d^2}{r^2} + d^2 = p^2 - \frac{d^2}{r^2} \times \overline{p^2 - r^2} = a^2$

& since $\therefore CT$ is not affected by r or d CT to any other distance will be equal to a. agreeably to Cotes –

To accommodate this to Cotes' Construction, we must take the corresponding sector of a circle whose radius is $\sqrt{QT^2 + CQ^2}$ or a

Now such sector $= \text{ours} \times \frac{a^2}{r^2} = \frac{1}{2} \frac{a \sqrt{p^2 - r^2} \times \pm \dot{x}}{x \sqrt{a^2 - x^2}}$; & arc $= \frac{a \sqrt{p^2 - r^2} \times \pm \dot{x}}{x \sqrt{a^2 - x^2}}$

as in Cotes by Saunderson; for his expression is $\frac{b n \dot{x}}{x \sqrt{n^2 - x^2}}$ where

n is the apse distance or our a, & $b^2 = p^2 - n^2 = \text{our } p^2 - a^2$.

Case 2. Let the vel^y be that acquired from an infinite distance.

Then area $= \frac{D}{2x^2}$; ~~&~~ $\therefore$ area $: \frac{Q^2}{x^2}$ in

a constant ratio viz. $\frac{D}{2} : Q^2 :: \frac{D}{2} : d^2 \frac{D}{2r^2} :: r^2 : d^2$

$\therefore x^2 : CQ^2 :: r^2 : d^2$ & $x : CQ :: r : d$ & the curve is the logarithmic spiral with all its properties.

If the body be projected from an apse i.e. if $r = d$

x will always be $=$ to CQ & the body will move in a $\odot$

Case 3^d. Let the vel at V be greater than that from an infinite distance — let it be that w^{ch} might be acquired at V by a body projected from infinite distance with velocity obtained ~~by~~ falling from infinite distance to distance p; i.e. let ~~the area at infinite distance~~ $\overline{pQRV - \text{area from V up ad infin}}$ $= \frac{D}{-2} \times -p^{-2}$; then the area $PQRV \pm VLFD = \frac{D}{-2} \times -x^{-2} - p^{-2}$

$= \frac{D}{2} \times \frac{p^2 + x^2}{p^2 x^2}$ hence $\frac{r^2 Q \times \pm \dot{x}}{2x^2 \sqrt{\text{area} - v^2}} = \frac{\frac{1}{2} r^2 Q \times \pm \dot{x}}{x^2 \sqrt{\frac{D}{2} \cdot \frac{p^2 + x^2}{p^2 x^2} - \frac{Q^2}{x^2}}}$

$= \frac{\frac{1}{2} r^2 Q \times \pm \dot{x}}{x \sqrt{\frac{D}{2 p^2} \times p^2 + x^2 - \frac{2 Q^2 p^2}{D}}} = \frac{\frac{\frac{1}{2} r^2 Q P}{\sqrt{\frac{D}{2}}} \times \pm \dot{x}}{x \sqrt{p^2 - \frac{2 Q^2 p^2}{D} + x^2}}$, which has 3

different fluents according as $p^2 - \frac{2Q^2 p^2}{D}$ is pos^{ve}, 0, or neg^{ve} i.e. according as Q^2 is less, equal or g^r than $\frac{D}{2}$.

Part 1st

1st let $p^2 - \frac{2Q^2 p^2}{D}$ be positive $= a^2$, then as in 1st case, $\frac{\frac{1}{2} r^2 Q P}{\sqrt{\frac{D}{2}}}$ $= \frac{1}{2} r^2 \sqrt{p^2 - a^2}$ & the fluxion becomes $\frac{\frac{1}{2} r^2 \sqrt{p^2 - a^2} \times \pm \dot{x}}{x \sqrt{a^2 + x^2}}$

$= \frac{\frac{1}{2} r^2 \sqrt{p^2 - a^2}}{2a} \times \frac{\pm 2a\dot{x}}{x\sqrt{a^2 + x^2}}$, whose fluent corrected is

$\frac{\frac{1}{2} r^2 \sqrt{p^2 - a^2}}{2a} \times \left(\text{h.l.} \frac{\sqrt{a^2 + x^2} - a}{a + \sqrt{a^2 + x^2}} \sim \text{h.l.} \frac{\sqrt{a^2 + r^2} - a}{a + \sqrt{a^2 + r^2}} \right)$ or the difference of the Logs of $\frac{\sqrt{a^2 + x^2} - a}{a + \sqrt{a^2 + x^2}}$ & $\frac{\sqrt{a^2 + r^2} - a}{a + \sqrt{a^2 + r^2}}$ to Mod. $\frac{\frac{1}{2} r^2 \sqrt{p^2 - a^2}}{2a}$

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or is $\frac{\frac{1}{2} r^2 \sqrt{p^2-a^2}}{2a} \pm \left| \frac{\sqrt{a^2+x^2} - a \times a + \sqrt{a^2+r^2}}{a + \sqrt{a^2+x^2} \times \sqrt{a^2-r^2} - a} \right.$ or is

$$\frac{\frac{1}{2} r^2 \sqrt{p^2-a^2}}{2a} \pm \left| \frac{\left(\sqrt{a^2+x^2} - a\right)^2 \times \left(a + \sqrt{a^2+r^2}\right)^2}{x^2 \times r^2} \right. = \frac{1}{2} \frac{r^2\sqrt{p^2-a^2}}{a} \pm \left| \frac{\sqrt{a^2+x^2} - a \times a + \sqrt{a^2+r^2}}{rx} \right.$$

the sector to rad. CV between CV and distance x

If we suppose x infinite, the expression becomes

$$\frac{1}{2} \frac{r^2 \sqrt{p^2-a^2}}{2a} \times h.l. \frac{\sqrt{a^2+r^2}-a}{a+\sqrt{a^2+r^2}} \quad \text{or} \quad \frac{\frac{1}{2} r^2 \sqrt{p^2+a^2}}{a} \left| \frac{\sqrt{a^2-r^2}-a}{r} \right.$$

a finite q.y; or the angle the body can describe in ascending from any point V is limited. Hence the sector between this limiting position of the ray CI, & any distance x is

$$\frac{1}{2} \frac{r^2 \sqrt{p^2-a^2}}{a} \left| \frac{\sqrt{a^2-x^2}-a}{x} \right. .$$

Part. 2d.

Let $p^2 - \frac{2a^2p^2}{D} = 0$; then $p^2 = \frac{2Qp^2}{D}$ $\therefore \frac{Qp}{\sqrt{\frac{D}{2}}} = p$

& fluxion becomes $\frac{\frac{1}{2} r^2 p \times \pm \dot{x}}{x\sqrt{x^2}} = \frac{1}{2} \frac{r^2 p \times \pm \dot{x}}{x^2}$, whose fluent corrected is $\frac{1}{2} r^2 p \times \frac{1}{x} \sim \frac{1}{r} = \frac{1}{2} r^2 p \times \frac{r \sim x}{rx}$.

hence If x be infinite $VCx = \frac{\frac{1}{2} r^2 p}{r} = \frac{1}{2} r p$; & the arc $Vx = p$, i.e. the ∠ that can be described by the body ascending from any point V is limited, or the curve has a radius primus, moreover the sector to any rad. CV between CV & radius primus is as CV, or the arc is constant $= p$.

Part 3

Let $p^2 - \frac{2Q^2p^2}{D}$ be negative $= -a^2$; then

$p^2 + a^2 = \frac{2Q^2p^2}{D}$ & $\frac{\frac{1}{2}rQp}{\sqrt{\frac{D}{2}}} = \frac{1}{2}r^2\sqrt{p^2+a^2}$; hence the

fluxion becomes $\frac{\frac{1}{2}r^2\sqrt{p^2+a^2}\times\pm\dot{x}}{x\sqrt{x^2-a^2}}$, whose fluent may be

expressd both by ~~elliptic sectors & by~~ ⊙r arcs & by Cir Sectors

corrected

By ~~⊙r arcs~~ the fluent is $\int\frac{1}{2}r^2\frac{\sqrt{p^2+a^2}}{a^2}\times\frac{\pm a^2\dot{x}}{x\sqrt{a^2+x^2}}+C =$

$\frac{1}{2}r^2\frac{\sqrt{p^2+a^2}}{a^2}\times$ (⊙r arc, rad. a, & secant x ∼ ⊙r arc rad. a & sec. r)

The body is at lower apse when $x = a$. & sector between apse distance & any distance CV, is $\frac{1}{2}r^2\frac{\sqrt{p^2+a^2}}{a^2}\times$ arc. sec$^+$ x − arc sec a

$= \frac{1}{2}r^2\frac{\sqrt{p^2+a^2}}{a^2}\times$ arc sec$^+$ x. & ∴ sector from apse to any distance x is $\frac{1}{2}r^2\frac{\sqrt{p^2+a^2}}{a^2}\times$ arc. sec$^+$ x.

If we suppose x infinite, the arc whose secant is x is a quadrant hence the sector to rad r between apse & limiting position of the ray is $\frac{1}{2}r^2\frac{\sqrt{p^2+a^2}}{a^2}\times$ Quad. of ⊙ whose rad is a.

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In the 3 last spirals if any ray CQ be produced, or (which is more convenient) be ~~till CT~~ to A so that $CA = p$ & $\perp^r$ AT be drawn on the tangent at Q

$QT^2 - CQ^2$ will be $=$ to $p^2 - \frac{2 a^2 p^2}{D}$

For $p^2 - \frac{2 a^2 p^2}{D} = p^2 - \frac{a^2}{r^2} \times \overline{p^2 + r^2} = p^2 - \frac{a^2 p^2}{r^2} - a^2$

Now by simi^r Δs $r^2 : r^2 - a^2 :: p^2 : QT^2 = \frac{p^2 r^2}{r^2} - \frac{a^2 p^2}{r^2}$

$\therefore QT^2 - CQ^2 = p^2 - \frac{a^2 p^2}{r^2} - a^2$

$p^2 - a^2$ in 1st part $= \frac{a^2}{r} \cdot \overline{p^2 + r^2}$; & $p^2 + a^2$ in 2d part $= \frac{a^2}{r} \times \overline{p^2 \mp r^2}$

To accommodate the construct^n to Cotes' take in the 1st part

$\dot{\text{sector}}$ of ⊙ whose radius is $\sqrt{QT^2 - CQ^2}$ or a. Now this sector

$=$ ours $\times \frac{a^2}{r^2}$ hence flux^n becomes $\frac{\frac{1}{2} a^2 \times \sqrt{p^2 - a^2} \times \pm \dot{x}}{x \sqrt{x^2 + a^2}}$; &

$\dot{\text{arc}} = \frac{a \sqrt{p^2 - a^2} \times \pm \dot{x}}{x \sqrt{a^2 + x^2}}$ as in Saunderson

For the 2d part the expressions agree

Take in the 3d part $\dot{\text{sector}}$ whose rad $= \sqrt{CQ^2 - QT^2} = a$

this sector $=$ ours $\times \frac{a^2}{r^2}$ hence flux^n becomes $\frac{\frac{1}{2} a^2 \sqrt{p^2 + a^2} \pm \dot{x}}{x \sqrt{x^2 + a^2}}$

& $\dot{\text{arc}} = \frac{a \sqrt{p^2 + a^2} \times \pm \dot{x}}{x \sqrt{x^2 - a^2}}$ as in Saunderson

Of vel in the 3 spirals compar'd with the vely in ⊙ at same distance

~~v^2 in 1st case~~ $\cancel{= \frac{D}{2} \times \frac{p^2-x^2}{p^2x^2} \times \frac{4s}{Dr^{n-1}} = \frac{p^2-x^2}{p^2x^2} \times 2sr^3}$

∴ v^2 in ⊙ $= \frac{2sx^n}{r^{n-1}} = \frac{2sr^3}{x^2}$

Now v^2 in 1st case $= \frac{D}{2} \frac{p^2-x^2}{p^2x^2} \times \frac{4s}{Dr^{n-1}} = \frac{p^2-x^2}{p^2x^2} \times 2sr^3$

∴ $v^2 : v^2$ in ⊙ $:: \frac{p^2-x^2}{p^2x^2} : \frac{1}{x^2} :: p^2-x^2 : p^2$

& $v : v$ in ⊙ $:: \sqrt{p^2-x^2} : p$

In 2d case $v^2 = \frac{D}{2x^2} \times \frac{4sr^3}{D} = \frac{2sr^3}{x^2} = v^2$ in ⊙

In ye 3 part of 3d case $v^2 = \frac{D}{2} \cdot \frac{p^2+x^2}{p^2x^2} \times \frac{4sr^3}{D} = \frac{p^2+x^2}{p^2x^2} \cdot 2sr^3$

∴ $v^2 : v^2$ in ⊙ $:: p^2+x^2 : p^2$

& $v : v$ in ⊙ $:: \sqrt{p^2+x^2} : p$

The same ~~than~~ proportions might be determind without considering the general expressn of v^2 in ⊙; by means of 2d case where it appears that v is = that in ⊙.